Cambridge Assessment
Admissions Testing

STEP Mark Schemes 2017

Mathematics

STEP 9465/9470/9475

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Introduction

These mark schemes are published as an aid for teachers and students, and indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches; if a different approach was taken by a candidate, their solution was marked accordingly after discussion by the marking team. These adaptations are not recorded here.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

Admissions Testing will not enter into any discussion or correspondence in connection with this mark scheme.

Marking notation

NOTATION	MEANING	NOTES
M	Method mark	For correct application of a **M**ethod.
dM or m	Dependent method mark	This cannot be earned unless the preceding **M** mark has been earned.
A	Answer mark	**M0** ⇒ **A0**
B	Independently earned mark	Stand alone for "right or wrong".
E	B mark for an explanation	
G	B mark for a graph	
ft	Follow through	To highlight where incorrect answers should be marked as if they were correct.
CAO or **CSO** **Sometimes written as A***	Correct Answer/Solution Only	To emphasise that **ft** does not apply.
AG	Answer Given	Indicates answer is given in question.

Question 1

(i) $\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r} = \frac{(n-1)!\,r!}{(n+r-1)!} - \frac{n!r!}{(n+r)!}$ **M1**

$= \frac{(n-1)!\,r!\,[(n+r)-n]}{(n+r)!}$

$= \frac{(n-1)!\,r!\,r}{(n+r)!}$ **M1**

$\therefore \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right) = \frac{r+1}{r}\frac{(n-1)!\;r!\;r}{(n+r)!}$

$= \frac{(n-1)!\,(r+1)!}{(n+r)!} = \frac{1}{{}^{n+r}C_{r+1}}$ **A1*** **(3)**

$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}} = \sum_{n=1}^{\infty} \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right)$ **M1**

$= \frac{r+1}{r}\left(\frac{1}{{}^{r}C_r} - \frac{1}{{}^{r+1}C_r} + \frac{1}{{}^{r+1}C_r} - \frac{1}{{}^{r+2}C_r} + \frac{1}{{}^{r+2}C_r} - \frac{1}{{}^{r+3}C_r} + \cdots\right)$ **M1**

$= \frac{r+1}{r}\frac{1}{{}^{r}C_r}$ because ${}^{n+r}C_r \to \infty$ as $n \to \infty$ **E1**

$= \frac{r+1}{r}$ **A1** **(4)**

$\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_{2+1}} = \frac{2+1}{2} - \frac{1}{{}^{1+2}C_{2+1}} = \frac{3}{2} - 1 = \frac{1}{2}$ **M1 M1 (2)**

(ii) ${}^{n+1}C_3 = \frac{(n+1)!}{(n-2)!3!} = \frac{(n+1)n(n-1)}{3!} = \frac{n^3-n}{3!} < \frac{n^3}{3!}$ **M1**

So $\frac{3!}{n^3} < \frac{1}{{}^{n+1}C_3}$ **A1*** **(2)**

$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3} = \frac{120}{n(n^2-1)} - \frac{120}{n(n^2-1)(n^2-4)} - \frac{120}{n^3}$

$= \frac{120}{n^3(n^2-1)(n^2-4)}\left(n^2(n^2-4) - n^2 - (n^2-1)(n^2-4)\right)$ **M1**

$= \frac{-480}{n^3(n^2-1)(n^2-4)} < 0$

as $n \geq 3$ and so denominator is positive. **E1** **(2)**

Hence, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} < \frac{5!}{n^3}$

Alternatively,

$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} = \frac{5!}{n(n^2-1)} - \frac{5!}{n(n^2-1)(n^2-4)} = \frac{5!}{n(n^2-1)(n^2-4)} \times \left((n^2-4)-1\right)$

$= \frac{5!}{n^3} \times \frac{n^4-5n^2}{n^4-5n^2+4} < \frac{5!}{n^3}$

as $n \geq 3$ and so $n^2 > 5$

$$\sum_{n=3}^{\infty} \frac{3!}{n^3} < \sum_{n=3}^{\infty} \frac{1}{{}^{n+1}C_3} = \sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_3} = \frac{1}{2}$$

M1

So $\sum_{n=1}^{\infty} \frac{3!}{n^3} < \frac{3!}{1} + \frac{3!}{8} + \frac{1}{2} = \frac{29}{4}$

M1

And therefore $\sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{29}{24} = \frac{116}{96}$

A1* (3)

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \sum_{n=3}^{\infty} \left(\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} \right) = 20 \times \frac{1}{2} - \left(\sum_{n=1}^{\infty} \frac{1}{{}^{n+4}C_5} \right) = 10 - \frac{5}{4}$$

M1 **M1**

Therefore

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \frac{35}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} > \frac{35}{4 \times 5!} + \frac{1}{1} + \frac{1}{8} = \frac{7}{96} + \frac{96}{96} + \frac{12}{96} = \frac{115}{96}$$

M1 **A1* (4)**

Question 2

(i) $z' - a = e^{i\theta}(z - a)$ **M1**

Thus $z' = a + e^{i\theta}z - e^{i\theta}a = e^{i\theta}z + a(1 - e^{i\theta})$ **A1* (2)**

(ii) $z'' = e^{i\varphi}z' + b(1 - e^{i\varphi})$

$= e^{i\varphi}\left(e^{i\theta}z + a(1 - e^{i\theta})\right) + b(1 - e^{i\varphi})$ **M1 A1**

So $z'' = e^{i(\varphi+\theta)}z + \left(ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}\right)$

This is a rotation about c if

$c\left(1 - e^{i(\varphi+\theta)}\right) = ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}$ **M1**

If $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found. **E1**

Otherwise, multiplying by $-e^{\frac{-i(\varphi+\theta)}{2}}$,

$c\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right) = a\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{i(\varphi-\theta)}{2}}\right) + b\left(e^{\frac{i(\varphi-\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right)$

$2ci\sin\frac{1}{2}(\varphi+\theta) = 2aie^{i\varphi/2}\sin\frac{1}{2}\theta + 2bie^{-i\theta/2}\sin\frac{1}{2}\varphi$ **M1**

$c\sin\frac{1}{2}(\varphi+\theta) = ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi$ **A1* (6)**

If $\varphi + \theta = 2n\pi$, $z'' = z + \left(ae^{i\varphi} - a + b - be^{i\varphi}\right)$ **M1**

So $z'' = z + (b - a)(1 - e^{i\varphi})$ **A1**

This is a translation by $(b - a)(1 - e^{i\varphi})$ **A1 (3)**

(iii) If RS = SR, and if $\varphi + \theta = 2n\pi$, then

$$(b - a)(1 - e^{i\varphi}) = (a - b)(1 - e^{i\theta})$$

M1

$$(a - b)(e^{i\theta} + e^{i\varphi} - 2) = 0$$

So $a = b$, or if $a \neq b$, $e^{i\theta} + e^{i(2n\pi-\theta)} - 2 = 0$

A1

$2\cos\theta - 2 = 0$ **M1**

Thus $\theta = 2n\pi$ **A1 (4)**

If $\varphi + \theta \neq 2n\pi$

$ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi = be^{i\theta/2}\sin\frac{1}{2}\varphi + ae^{-i\varphi/2}\sin\frac{1}{2}\theta$ **M1**

$2i(a - b)\sin\frac{1}{2}\varphi\sin\frac{1}{2}\theta = 0$ **A1**

So $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

A1 **A1** **A1 (5)**

Question 3

$\alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = -A$ **M1**

$A = -q$ **A1 (2)**

(i) $y^3 - 3y^2 - 40y + 84 = 0$ **M1 A1**

$(y-2)(y^2 - y - 42) = 0$ **M1**

$(y-2)(y-7)(y+6) = 0$ **M1 A1**

So $\alpha\beta + \gamma\delta = 7$ **A1 (6)**

(ii) $(\alpha+\beta)(\gamma+\delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = 3 - \alpha\beta - \gamma\delta = -4$

M1 **M1** **A1 (3)**

$(\alpha+\beta) + (\gamma+\delta) = 0$ **M1**

Thus $(\alpha+\beta)$ is a root of $t^2 - 4 = 0$ **M1**

So $\alpha + \beta = \pm 2$ **A1**

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 6$$

$$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = 6$$

$2(\alpha\beta - \gamma\delta) = \pm 6$ **M1**

$\alpha\beta - \gamma\delta = 3$ as $\alpha\beta > \gamma\delta$ (and so $\alpha + \beta = -2$)

So $\alpha\beta = 5$ **A1 (5)**

Alternatively, $\alpha\beta\gamma\delta = 10$, **M1 A1** so $\alpha\beta$ and $\gamma\delta$ are the roots of

$t^2 - 7t + 10 = 0$ **M1 A1** and as $\alpha\beta > \gamma\delta$, $\alpha\beta = 5$ (and $\gamma\delta = 2$). **A1 (5)**

(iii) Thus α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of $t^2 - 2t + 2 = 0$ **M1 A1**

So $x = 1 \pm i$, $-1 \pm 2i$ **A1 A1 (4)**

Question 4

(i) $e^{x\ln a} = a^x$ (formula book)

So if $\log_a f(x) = z$

$f(x) = a^z = e^{z\ln a}$ **E1**

and so $\ln f(x) = z\ln a = \ln a\ \log_a f(x)$ **B1**

Therefore,

$$e^{\frac{1}{y}\int_0^y \ln f(x)dx} = e^{\frac{1}{y}\int_0^y \ln a\ \log_a f(x)\,dx} = e^{\frac{1}{y}\ln a\int_0^y \log_a f(x)\,dx}$$

M1 **M1**

Thus, $F(y) = a^{\frac{1}{y}\int_0^y \log_a f(x)\,dx}$ **A1* (5)**

(ii) $H(y) = e^{\frac{1}{y}\int_0^y \ln h(x)dx} = e^{\frac{1}{y}\int_0^y \ln(f(x)g(x))dx}$ **M1**

$$= e^{\frac{1}{y}\int_0^y \ln f(x)+\ln g(x)dx}$$

$= e^{\frac{1}{y}\left(\int_0^y \ln f(x)dx+\int_0^y \ln g(x)dx\right)}$ **M1**

$= e^{\frac{1}{y}\int_0^y \ln f(x)dx}\ e^{\frac{1}{y}\int_0^y \ln g(x)dx} = F(y)G(y)$

M1 **A1* (4)**

(iii) Let $f(x) = b^x$,

Then $F(y) = e^{\frac{1}{y}\int_0^y \ln b^x dx} = e^{\frac{1}{y}\int_0^y x\ln b\ dx} = e^{\frac{1}{y}\ln b\ \int_0^y x dx}$

M1 **M1**

$= e^{\frac{1}{y}\ln b\ \left[\frac{1}{2}x^2\right]_0^y} = e^{\frac{1}{y}\ln b\ \frac{1}{2}y^2} = e^{\frac{1}{2}y\ln b} = b^{\frac{1}{2}y} = \sqrt{b^y}$

A1 **M1** **A1* (5)**

(iv) $e^{\frac{1}{y}\int_0^y \ln f(x)dx} = \sqrt{f(y)}$

$\frac{1}{y}\int_0^y \ln f(x)dx = \ln\sqrt{f(y)} = \frac{1}{2}\ln f(y)$

$\int_0^y \ln f(x)dx = \frac{y}{2}\ln f(y)$ **M1**

$\ln f(y) = \frac{1}{2}\ln f(y) + \frac{yf'(y)}{2f(y)}$ **M1**

$\frac{yf'(y)}{f(y)} = \ln f(y)$ so $\frac{f'(y)}{f(y)\ln f(y)} = \frac{1}{y}$ **M1**

Integrating $\ln\ln f(y) = \ln y + c = \ln y + \ln k = \ln ky$ **M1 A1**

$\ln f(y) = ky$

$f(y) = e^{ky} = e^{y\ln b} = b^y$

$f(x) = b^x$ **A1* (6)**

Question 5

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta = f\cos\theta + f'\sin\theta$ **M1**

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta = -f\sin\theta + f'\cos\theta$ **M1**

$\frac{dy}{dx} = \frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta} = \frac{f + f'\tan\theta}{-f\tan\theta + f'}$ **M1 A1 (4)**

$\frac{f + f'\tan\theta}{-f\tan\theta + f'} \times \frac{g + g'\tan\theta}{-g\tan\theta + g'} = -1$ **M1**

$fg + f'g\tan\theta + fg'\tan\theta + f'g'\tan^2\theta = -fg\tan^2\theta + f'g\tan\theta + fg'\tan\theta - f'g'$

$(fg + f'g')\sec^2\theta = 0$ **M1**

$fg + f'g' = 0$ **A1* (3)**

$g(\theta) = a(1 + \sin\theta)$

$g'(\theta) = a\cos\theta$

So $f'a\cos\theta + fa(1 + \sin\theta) = 0$ **M1**

$\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\sec\theta - \tan\theta$ **A1**

$\ln f = -\ln(\sec\theta + \tan\theta) + \ln\cos\theta + c = \ln\left(\frac{k\cos\theta}{\sec\theta + \tan\theta}\right) = \ln\left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ **M1 A1**

$f(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right) = \frac{k(1-\sin^2\theta)}{1+\sin\theta} = k(1 - \sin\theta)$ **M1 A1)**

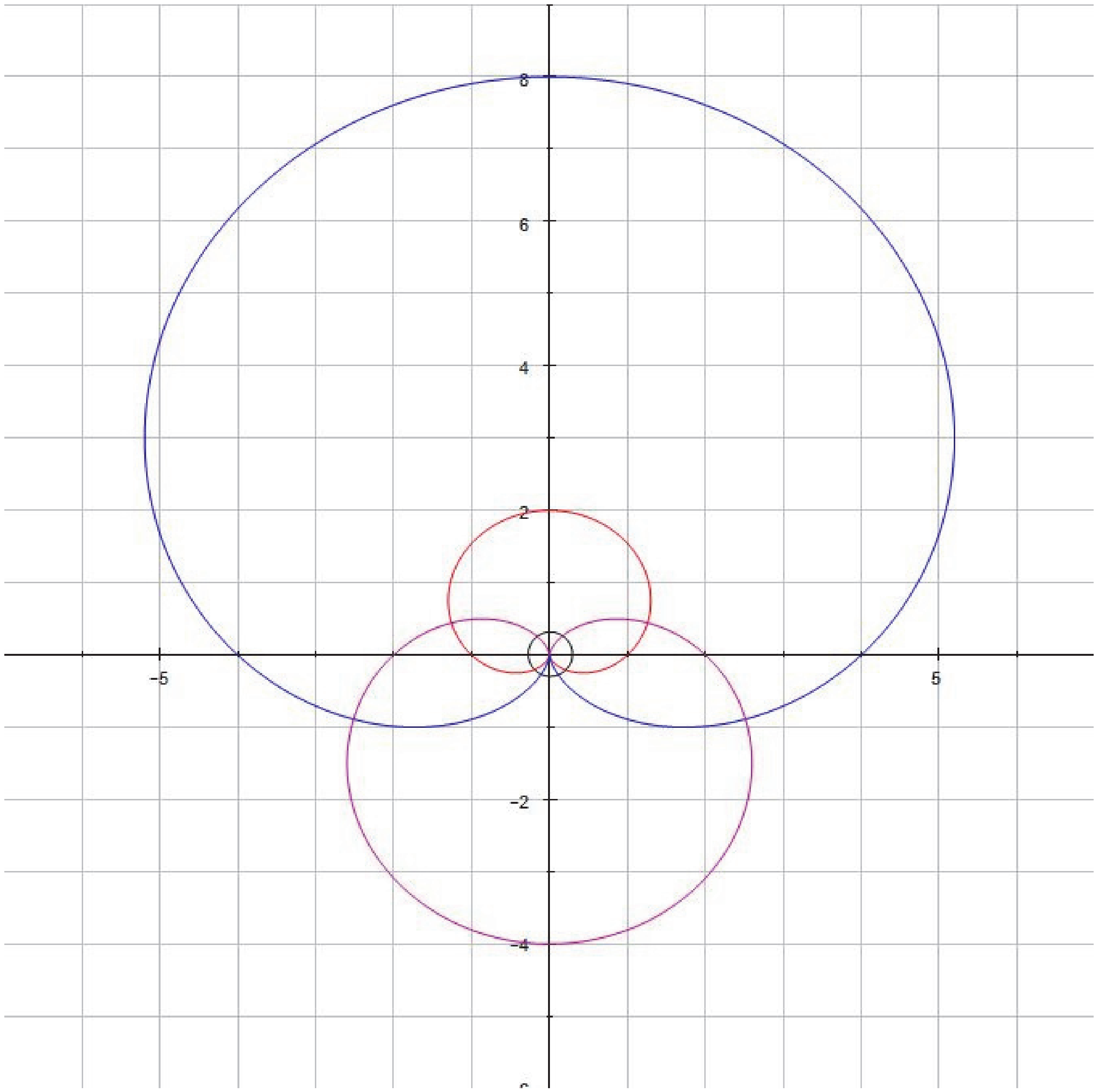
Alternatively, $\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\frac{\cos\theta}{(1-\sin\theta)}$ **M1 A1**

$\ln f = \ln\big((1 - \sin\theta)\big) + c = \ln\big(k(1 - \sin\theta)\big)$ **M1**

and hence $f(\theta) = k(1 - \sin\theta)$ **A1**

$r = 4, \theta = -\frac{1}{2}\pi$ so $4 = 2k$ **M1**

Thus $f(\theta) = 2(1 - \sin\theta)$ **A1 (8)**



G1 G1 dG1 G1 G1(5)

Question 6

(i) $T(x) = \int_0^x \frac{1}{1+u^2} du$

Let $u = v^{-1}$, $\frac{du}{dv} = -v^{-2}$

B1

So

$$T(x) = \int_{\infty}^{x^{-1}} \frac{1}{1+v^{-2}} \times -v^{-2} dv = \int_{x^{-1}}^{\infty} \frac{1}{v^2+1} dv = \int_0^{\infty} \frac{1}{1+u^2} du - \int_0^{x^{-1}} \frac{1}{1+u^2} du$$

M1 **M1**

$T(x) = T(\infty) - T(x^{-1})$ **A1* (4)**

(ii) $v = \frac{u+a}{1-au} \Leftrightarrow v - auv = u + a \Leftrightarrow v - u = a(1+uv) \Leftrightarrow a = \frac{v-u}{1+uv}$

M1

$0 = \frac{(1+uv)\left(\frac{dv}{du}-1\right)-(v-u)\left(u\frac{dv}{du}+v\right)}{(1+uv)^2}$ **M1**

$\frac{dv}{du}(1 + uv - uv + u^2) = 1 + uv + v^2 - uv$

$\frac{dv}{du} = \frac{1+v^2}{1+u^2}$ **A1* (3)**

Alternatively,

$$v = \frac{u+a}{1-au} \Leftrightarrow \frac{dv}{du} = \frac{(1-au)+a(u+a)}{(1-au)^2} = \frac{1+a^2}{(1-au)^2} = \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)}$$

M1

$$= \frac{(1-au)^2+(u+a)^2}{(1-au)^2(1+u^2)} = \frac{1+v^2}{1+u^2}$$

M1 **A1**

$$T(x) = \int_0^x \frac{1}{1+u^2} du = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+u^2}\frac{1+u^2}{1+v^2} dv = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv = \int_0^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv - \int_0^a \frac{1}{1+v^2} dv$$

M1 **M1**

$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ **A1* (3)**

As $T(x) = T(\infty) - T(x^{-1})$, $T(a) = T(\infty) - T(a^{-1})$

So

$$T(x^{-1}) = T(\infty) - T(x) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - T(a)\right) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - \left(T(\infty) - T(a^{-1})\right)\right)$$

M1 **M1**

Thus

$T(x^{-1}) = 2T(\infty) - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$ **A1* (3)**

Let $y = x^{-1}$, $= a^{-1}$, then $x < \frac{1}{a}$ implies $\frac{1}{y} < b$ which is $y > \frac{1}{b}$ **M1**

$T(y) = 2T(\infty) - T\left(\frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}}\right) - T(b) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ **A1* (2)**

(iii) Using $T(y) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ with $y = b = \sqrt{3}$ **M1**

$T(\sqrt{3}) = 2T(\infty) - T\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}\sqrt{3}-1}\right) - T(\sqrt{3})$

$T(\sqrt{3}) = 2T(\infty) - T(\sqrt{3}) - T(\sqrt{3})$

$3T(\sqrt{3}) = 2T(\infty) \quad \Leftrightarrow \quad T(\sqrt{3}) = \frac{2}{3}T(\infty)$ **A1* (2)**

Using $T(x) = T(\infty) - T(x^{-1})$ with $x = 1$,

$T(1) = T(\infty) - T(1)$ and so $T(1) = \frac{1}{2}T(\infty)$ **B1**

Using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = \sqrt{2} - 1$ and $a = 1$ **M1**

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}-1\ +1}{1-(\sqrt{2}-1\ )}\right) - T(1)$

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right) - T(1) = T\left(\frac{1}{\sqrt{2}-1}\right) - T(1) = T(\sqrt{2}+1) - T(1)$

Using $T(x) = T(\infty) - T(x^{-1})$, $T(\sqrt{2} + 1\ ) = T(\infty) - T(\sqrt{2} - 1)$

So $T(\sqrt{2} - 1\ ) = T(\infty) - T(\sqrt{2} - 1) - T(1)$

$2T(\sqrt{2} - 1\ ) = T(\infty) - T(1) = T(\infty) - \frac{1}{2}T(\infty)$

$T(\sqrt{2} - 1\ ) = \frac{1}{4}T(\infty)$ **A1* (3)**

Alternatively, using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = a = \sqrt{2} - 1$

$$T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{1 - (\sqrt{2} - 1\ )^2}\right) - T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{2(\sqrt{2} - 1\ )}\right) - T(\sqrt{2} - 1\ )$$

Therefore $2T(\sqrt{2} - 1\ ) = T(1)$ and so $T(\sqrt{2} - 1\ ) = \frac{1}{2}T(1) = \frac{1}{4}T(\infty)$

Question 7

$$\frac{\frac{a^2(1-t^2)^2}{(1+t^2)^2}}{a^2}+\frac{\frac{4b^2t^2}{(1+t^2)^2}}{b^2}=\frac{(1-t^2)^2+4t^2}{(1+t^2)^2}=\frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}=1$$ **B1 (1)**

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0$ **M1**

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2a(1-t^2)(1+t^2)}{a^2(1+t^2)2bt}=-\frac{b(1-t^2)}{2at}$ **M1 A1**

So *L* is $y-\frac{2bt}{(1+t^2)}=-\frac{b(1-t^2)}{2at}\left(x-\frac{a(1-t^2)}{(1+t^2)}\right)$ **M1**

$2at(1+t^2)y-4abt^2=-bx(1-t^2)(1+t^2)+ab(1-t^2)^2$

$2at(1+t^2)y+bx(1-t^2)(1+t^2)=ab(1-t^2)^2+4abt^2=ab(1+t^2)^2$

Thus $2aty+bx(1-t^2)=ab(1+t^2)$ **M1**

and as (X,Y) lies on this line $2atY+bX(1-t^2)=ab(1+t^2)$

$0=(a+X)bt^2-2atY+b(a-X)$ **A1* (6)**

For there to be two distinct lines, there need to be two values of t .

So the discriminant must be positive, $(-2aY)^2-4(a+X)bb(a-X)>0$ **M1**

$4a^2Y^2>4b^2(a^2-X^2)$

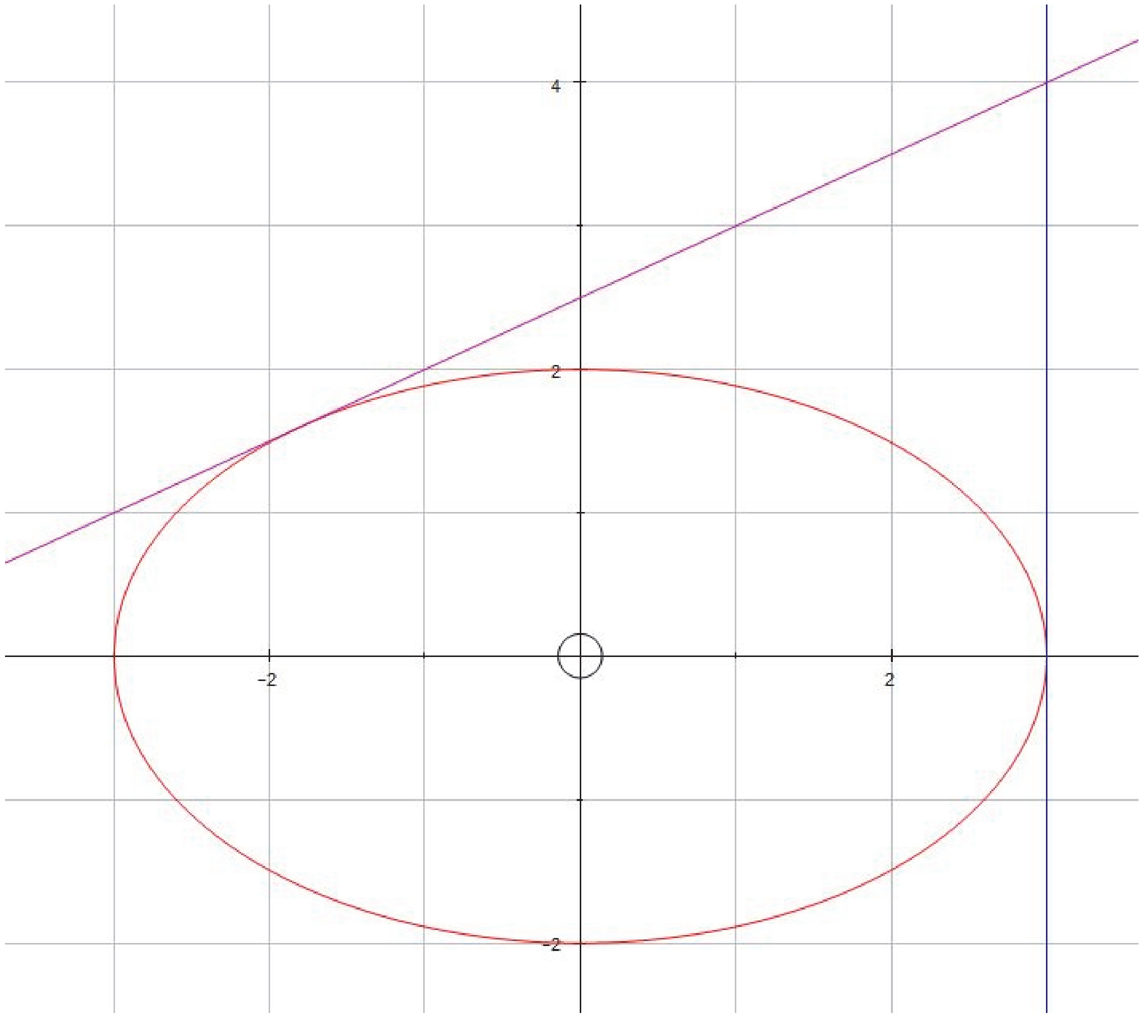
$a^2Y^2>(a^2-X^2)b^2$ **A1***

$\frac{Y^2}{b^2}>1-\frac{X^2}{a^2}$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$ so (X,Y) lies outside the ellipse. **B1 (3)**

However, if $X^2=a^2$, $=\pm a$, one tangent is at $t=0$ or $t=\infty$, a vertical line. **E1**

If $\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$, then $Y\neq 0$. **E1**



G1 (3)

(ii) p and q are the roots of $0 = (a+X)bt^2 - 2atY + b(a-X)$

So $p + q = \frac{2aY}{(a+X)b}$ and $pq = \frac{b(a-X)}{(a+X)b}$ **M1**

Thus $(a+X)pq = a - X$ and $(a+X)(p+q)b = 2aY$ **A1 A1 (3)**

Without loss of generality $(0, y_1)$ lies on $(a+x)bp^2 - 2apy + b(a-x) = 0$

and $(0, y_2)$ lies on $(a+x)bq^2 - 2aqy + b(a-x) = 0$

So $abp^2 - 2apy_1 + ab = 0$, that is $bp^2 - 2py_1 + b = 0$ **M1**

and $bq^2 - 2qy_2 + b = 0$

As $y_1 + y_2 = 2b$, $\frac{bp^2+b}{2p} + \frac{bq^2+b}{2q} = 2b$ **M1**

$\frac{p^2+1}{p} + \frac{q^2+1}{q} = 4$

$p + q + \frac{p+q}{pq} = 4$

$\frac{2aY}{(a+X)b} + \frac{\frac{2aY}{(a+X)b}}{\frac{a-X}{a+X}} = 4$ **M1**

$\frac{2aY}{a+X} + \frac{2aY}{a-X} = 4b$

$2aY(a - X + a + X) = 4(a - X)(a + X)b$

$4a^2Y = 4(a^2 - X^2)b$

$\frac{Y}{b} = 1 - \frac{X^2}{a^2}$

$\frac{X^2}{a^2} + \frac{Y}{b} = 1$ **A1* (4)**

Question 8

$$\sum_{m=1}^{n} a_m(b_{m+1}-b_m)+\sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m)$$
$$=\sum_{m=1}^{n}(a_m b_{m+1}-a_m b_m+b_{m+1}a_{m+1}-b_{m+1}a_m)$$

M1

$$=\sum_{m=1}^{n}(-a_m b_m+b_{m+1}a_{m+1})=a_{n+1}b_{n+1}-a_1 b_1$$

M1

Hence,

$$\sum_{m=1}^{n} a_m(b_{m+1}-b_m)=a_{n+1}b_{n+1}-a_1 b_1-\sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m)$$

A1* (3)

(i) Let $a_m=1$ (or any constant) and $b_m=\sin mx$, **M1**

then

$$\sum_{m=1}^{n}(\sin(m+1)x-\sin mx)=\sin(n+1)x-\sin x-\sum_{m=1}^{n}\sin(m+1)x\quad(1-1)$$

M1 A1

So

$$\sum_{m=1}^{n} 2\cos\left(m+\frac{1}{2}\right)x\ \sin\frac{1}{2}x=\ (\sin(n+1)x-\sin x)$$

M1 A1

and therefore

$$\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x=\frac{1}{2}(\sin(n+1)x-\sin x)\csc\frac{1}{2}x$$

A1* (6)

(ii) Let $a_m=m$ and $b_m=\sin(m-1)x-\sin mx$, **M1**

then

$$b_{m+1}-b_m=(\sin mx-\sin(m+1)x)-(\sin(m-1)x-\sin mx)$$
$$=-2\cos\left(m+\frac{1}{2}\right)x\ \sin\frac{1}{2}x+\ 2\cos\left(m-\frac{1}{2}\right)x\ \sin\frac{1}{2}x$$

M1 A1

$= 4 \sin mx \ \sin\frac{1}{2}x \sin\frac{1}{2}x$ **M1 A1**

Thus, using the stem

$$\sum_{m=1}^{n} m \times 4 \sin mx \ \sin^2 \frac{1}{2}x$$
$$= (n+1)(\sin nx - \sin(n+1)x) - 1 \times (\sin(0 \times x) - \sin x) - \sum_{m=1}^{n} (\sin mx - \sin(m+1)x)$$

M1 A1

So

$$4 \sin^2 \frac{1}{2}x \sum_{m=1}^{n} m \sin mx \ = (n+1)(\sin nx - \sin(n+1)x) + \sin x - \sin x + \sin(n+1)x$$

M1 A1

$$4 \sin^2 \frac{1}{2}x \sum_{m=1}^{n} m \sin mx \ = (n+1) \sin nx - n \sin(n+1)x$$

Thus

$$\sum_{m=1}^{n} m \sin mx \ = (p \sin nx + q \sin(n+1)x) \csc^2 \frac{1}{2}x$$

where

$$p = -\frac{1}{4}n$$

A1

and

$$q = \frac{1}{4}(n+1)$$

A1 (11)

Alternatively, let $a_m = m$ and $b_m = \cos\left(m - \frac{1}{2}\right)x$, using stem, **M1**

$$\sum_{m=1}^{n} m \left(\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right)$$
$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x$$

M1 A1

So,

$$\sum_{m=1}^{n} -2m \sin mx \ \sin\frac{1}{2}x$$

$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

M1 A1

$$= \csc\frac{1}{2}x\left((n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - \sin\frac{1}{2}x\cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\left(2(n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - 2\sin\frac{1}{2}x\cos\frac{1}{2}x - (\sin(n+1)x - \sin x)\right)$$

$$= \frac{1}{2}\csc\frac{1}{2}x\left((n+1)(\sin(n+1)x - \sin nx) - \sin x - \sin(n+1)x + \sin x\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\,(n\sin(n+1)x - (n+1)\sin nx)$$

giving result as before.

Question 9

For A, $mg - Z = m\ddot{y}$ and for B, $Z = 2m\ddot{x}$ where Z is tension. **M1 A1 A1**

Adding, $\ddot{y} + 2\ddot{x} = g$ **M1**

Integrating with respect to time, $\dot{y} + 2\dot{x} = gt + c$

Initially, $t = 0$, $\dot{x} = 0$, $\dot{y} = 0 \Rightarrow c = 0$

Integrating with respect to time, $y + 2x = \frac{1}{2}gt^2 + c'$ **M1 M1**

Initially, $t = 0$, $x = 0$, $y = 0 \Rightarrow c' = 0$

So $y + 2x = \frac{1}{2}gt^2$ **A1* (7)**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$ so $y = a$ **M1 A1**

Conserving energy, at time T we have shown there is no elastic potential energy, so

$$0 = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mga$$

M1 A1 A1 A1 (6)

That is

$$2\dot{x}^2 + \dot{y}^2 = 2ga$$

B1

But also $\dot{y} + 2\dot{x} = gT$ and so $\dot{y} + 2\dot{x} = \sqrt{6ga}$ **M1 A1**

Thus $2\dot{x}^2 + \left(\sqrt{6ga} - 2\dot{x}\right)^2 = 2ga$ **M1 A1**

$$6\dot{x}^2 - 4\dot{x}\sqrt{6ga} + 4ga = 0$$

$$\dot{x}^2 - 2\dot{x}\sqrt{\frac{2ga}{3}} + \frac{2ga}{3} = 0$$

$$\left(\dot{x} - \sqrt{\frac{2ga}{3}}\right)^2 = 0$$

M1

and so $\dot{x} = \sqrt{\frac{2ga}{3}}$ **A1* (7)**

Alternatively,

$$Z = \frac{\lambda(y - x)}{a}$$

M1

Subtracting,

$$2mg - 3Z = 2m(\ddot{y} - \ddot{x})$$

$$\ddot{y} - \ddot{x} = -\frac{3\lambda(y - x)}{2ma} + g$$

M1

So,

$$y - x = \frac{2mga}{3\lambda}(1 - \cos\omega t)$$

M1

where

$$\omega^2 = \frac{3\lambda}{2ma}$$

As $y + 2x = \frac{1}{2}gt^2$, $3x = \frac{1}{2}gt^2 - \frac{2mga}{3\lambda}(1 - \cos\omega t)$ **M1**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$

so $3a = 3a - \frac{2mga}{3\lambda}\left(1 - \cos\omega\sqrt{\frac{6a}{g}}\right)$ and thus $\frac{3\lambda}{2ma}\frac{6a}{g} = 4n^2\pi^2$, $\lambda = \frac{4n^2\pi^2 mg}{9}$ **M1**

$$3\dot{x} = gt - \frac{2mga\omega\sin\omega t}{3\lambda} = g\sqrt{\frac{6a}{g}} - 0$$

$$\dot{x} = \sqrt{\frac{2ga}{3}}$$

M1 A1* (7)

Question 10

Moment of inertia of PQ about axis through P is $\frac{1}{3}m(3a)^2 = 3ma^2$ **B1**

Conserving energy, $0 = \frac{1}{2}3ma^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mg\frac{3}{2}a\sin\theta - mgl\sin\theta$ **M1 A1 A1 A1**

Thus $(3a^2 + l^2)\dot{\theta}^2 = g(3a + 2l)\sin\theta$ **A1* (6)**

Differentiating with respect to time,

$$2(3a^2 + l^2)\dot{\theta}\ddot{\theta} = g(3a + 2l)\cos\theta\dot{\theta}$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Alternatively, taking moments about axis through P

$$m(3a^2 + l^2)\ddot{\theta} = mg\left(\frac{3}{2}a + l\right)\cos\theta$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Resolving perpendicular to the rod for the particle,

$$mg\cos\theta - R = ml\ddot{\theta}$$

M1 A1

Thus

$$R = mg\cos\theta - ml\ddot{\theta} = mg\cos\theta\left(1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)}\right)$$

M1 A1

$$1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)} = \frac{6a^2 + 2l^2 - 3al - 2l^2}{2(3a^2 + l^2)} = \frac{3a(2a - l)}{2(3a^2 + l^2)} > 0$$

because $l < 2a$ **A1 (5)**

Resolving along the rod towards P for the particle,

$$F - mg\sin\theta = ml\dot{\theta}^2$$

M1 A1

Thus

$$F = mg\sin\theta + ml\dot{\theta}^2 = mg\sin\theta\left(1 + \frac{l(3a+2l)}{(3a^2+l^2)}\right) = mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right)$$

M1

On the point of slipping $F = \mu R$, so **B1**

$$mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right) = \mu mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$$

Thus

$$\tan\theta = \frac{\mu a(2a-l)}{2(a^2+al+l^2)}$$

A1* (5)

At the instant of release, the equation of rotational motion for the rod ignoring the particle is

$$mg\frac{3a}{2} = 3ma^2\ddot{\theta}$$

and thus

$$\ddot{\theta} = \frac{g}{2a}$$

M1

Therefore the acceleration of the point on the rod where the particle rests equals $l\ddot{\theta} = \frac{lg}{2a} > g$ if $l > 2a$, and so the rod drops away from the particle faster than the particle accelerates and the particle immediately loses contact. **A1 (2)**

(Alternatively, for particle to accelerate with rod from previous working $R < 0$, **M1** meaning that it would have to be attached to so accelerate, and as it is only placed on the rod, this cannot happen.) **A1 (2)**

Question 11

(i) Conserving (linear) momentum

$$Mu - nmv = 0$$

M1

$$u = \frac{nmv}{M}$$

A1

$$K = \frac{1}{2}Mu^2 + n \times \frac{1}{2}mv^2 = \frac{1}{2}M\left(\frac{nmv}{M}\right)^2 + \frac{1}{2}nmv^2 = \frac{1}{2}nmv^2\left(\frac{nm}{M} + 1\right)$$

M1 **M1** **A1* (5)**

as required.

(ii) Conserving momentum before and after r th gun fired

$$\big(M + \big(n - (r-1)\big)m\big)u_{r-1} = (M + (n-r)m)u_r - m(v - u_{r-1})$$

M1 A1

Therefore

$$(M + (n-r)m)(u_r - u_{r-1}) = mv$$

M1

and so

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m}$$

A1* (4)

Summing this result for $r = 1$ to $r = n$,

$$u_n - u_0 = \frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m}$$

M1

Because

$$0 \leq n - r \leq n - 1$$

$$M \leq M + (n-r)m \leq M + (n-1)m$$

$$\frac{mv}{M + (n-1)m} \leq \frac{mv}{M + (n-r)m} \leq \frac{mv}{M}$$

with equality only for the term $r = n$

Thus

$$\frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m} < \frac{nmv}{M}$$

E1

As $u_0 = 0$, $u_n < \frac{nmv}{M} = u$ **A1* (3)**

(iii) Considering the energy of the truck and the $\big(n-(r-1)\big)$ projectiles before and after the r^{th} projectile is fired (the other $(r-1)$ already fired do not change their kinetic energy at this time),

$$K_r - K_{r-1} = \frac{1}{2}(M+(n-r)m){u_r}^2 + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}\big(M+\big(n-(r-1)\big)m\big){u_{r-1}}^2$$

M1 A1

$$= \frac{1}{2}(M+(n-r)m)({u_r}^2 - {u_{r-1}}^2) + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}m{u_{r-1}}^2$$

$$= \frac{1}{2}(M+(n-r)m)(u_r - u_{r-1})(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

M1

Summing this result for $r = 1$ to $r = n$,

$$K_n - K_0 = \frac{1}{2}nmv^2 + \frac{1}{2}mv(u_n - u_0)$$

M1

So

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n$$

A1* (5)

Now

$$u_n < \frac{nmv}{M}$$

so

$$\frac{1}{2}mvu_n < \frac{1}{2}\frac{nm^2v^2}{M}$$

M1

and thus

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n < \frac{1}{2}nmv^2 + \frac{1}{2}\frac{nm^2v^2}{M} = \frac{1}{2}nmv^2\left(1+\frac{m}{M}\right) < \frac{1}{2}nmv^2\left(\frac{nm}{M}+1\right)$$
$$= K$$

M1

as $n > 1$ **E1 (3)**

Question 12

(i)

$$\sum_{y=1}^{n}\sum_{x=1}^{n} P(X=x, Y=y) = 1$$

$$\sum_{y=1}^{n}\sum_{x=1}^{n} k(x+y) = 1$$

M1

$$k\sum_{y=1}^{n}\left(\frac{1}{2}n(n+1) + ny\right) = 1$$

M1 A1

$$k\left(\frac{1}{2}n^2(n+1) + \frac{1}{2}n^2(n+1)\right) = 1$$

M1

Therefore,

$$k = \frac{1}{n^2(n+1)}$$

A1 (5)

$$P(X=x) = \sum_{y=1}^{n} k(x+y) = k\left(nx + \frac{1}{2}n(n+1)\right) = \frac{\left(2nx + n(n+1)\right)}{2n^2(n+1)} = \frac{n+1+2x}{2n(n+1)}$$

M1 A1 (2)

$$P(Y=y) = \frac{n+1+2y}{2n(n+1)}$$

B1

For X and Y to be independent, $P(X=x, Y=y) = P(X=x) \times P(Y=y)$ **M1**

So

$$\frac{n+1+2x}{2n(n+1)} \times \frac{n+1+2y}{2n(n+1)} = \frac{(x+y)}{n^2(n+1)}$$

M1

$$(n+1+2x)(n+1+2y) = 4(n+1)(x+y)$$

$$(n+1)^2 - 2(n+1)(x+y) + 4xy = 0$$

$$\left((n+1) - (x+y)\right)^2 - (x-y)^2 = 0$$

M1

which does not happen for e.g. $x = n$, $y = 1$. (Many equally valid examples possible.)

X and Y are not independent. **E1 (5)**

(ii)

$$E(XY)=\sum_{y=1}^{n}\sum_{x=1}^{n}kxy(x+y)=k\sum_{y=1}^{n}\left(y\frac{n(n+1)(2n+1)}{6}+y^2\frac{n(n+1)}{2}\right)$$

M1

$$=k\frac{n^2(n+1)^2(2n+1)}{6}=\frac{(n+1)(2n+1)}{6}$$

M1 A1 (3)

$$E(X)=E(Y)=\sum_{x=1}^{n}x\frac{n+1+2x}{2n(n+1)}=\frac{\frac{1}{2}n(n+1)^2+\frac{2}{6}n(n+1)(2n+1)}{2n(n+1)}$$

M1 A1

$$=\frac{(n+1)}{4}+\frac{(2n+1)}{6}=\frac{(7n+5)}{12}$$

A1 (3)

Thus

$$Cov(X,Y)=\frac{(n+1)(2n+1)}{6}-\left(\frac{(7n+5)}{12}\right)^2=\frac{-n^2+2n-1}{144}=\frac{-(n-1)^2}{144}<0$$

M1 **E1 (2)**

Question 13

$$V(x) = E((X-x)^2) = E(X^2) - 2xE(X) + x^2 = \sigma^2 + \mu^2 - 2x\mu + x^2 = \sigma^2 + (x-\mu)^2$$

M1 M1 M1 A1 (4)

$$E(Y) = E\big(V(X)\big) = E(\sigma^2 + (X-\mu)^2) = \sigma^2 + \sigma^2 = 2\sigma^2$$

M1 A1* (2)

If $X \sim U(0,1)$, then $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$, so $V(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{3}$

B1 B1 M1 A1 (4)

$$Y = V(X) = X^2 - X + \frac{1}{3} = \frac{1}{12} + \left(X - \frac{1}{2}\right)^2$$

$$Y \in \left[\frac{1}{12}, \frac{1}{3}\right]$$

$$P(Y < y) = P\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right) = P\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} < X < \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right)$$

$$= 2\sqrt{y - \frac{1}{12}}$$

M1 M1 A1

$f(y) = \frac{d}{dy}\big(F(y)\big) = \frac{d}{dy}\left(2\sqrt{y - \frac{1}{12}}\right) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \le y \le \frac{1}{3}$ and 0 otherwise.

M1 A1 A1 (6)

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{\frac{1}{12}}^{\frac{1}{3}} \left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy$$

M1 M1

$$= \left[\frac{2}{3}\left(y - \frac{1}{12}\right)^{\frac{3}{2}} + \frac{1}{6}\left(y - \frac{1}{12}\right)^{\frac{1}{2}}\right]_{\frac{1}{12}}^{\frac{1}{3}} = \frac{1}{12} + \frac{1}{12} = 2 \times \frac{1}{12}$$

M1 A1 (4)

as required.

Alternatively, for final integral,

let $u^2 = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{0}^{\frac{1}{2}} \frac{u^2 + \frac{1}{12}}{u} \, 2udu = \left[\frac{2}{3}u^3 + \frac{1}{6}u\right]_0^{\frac{1}{2}} = 2 \times \frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**

or further

let $u = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{0}^{\frac{1}{4}} \frac{u + \frac{1}{12}}{u^{\frac{1}{2}}} \, du = \left[\frac{2}{3}u^{\frac{3}{2}} + \frac{1}{6}u^{\frac{1}{2}}\right]_0^{\frac{1}{4}} = 2 \times \frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**